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The Limit of Mean Transformed Rational Approximation on Subsets

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Let X be a compact subset of Euclidean l-space and let $\int$ be the integral on X. Let τ be a continuous function from the real line into the nonnegative real line. For g measurable on X, consider the "τ-norm"

$$N(g) = \int \tau(g).$$

Let $\{\phi_1, ..., \phi_n\}$, $\{\psi_1, ..., \psi_m\}$ be linearly independent sequences of real functions on X. Define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let σ be a continuous mapping of the real line into the extended real line. Define

$$F(A, x) = \sigma(R(A, x)).$$

The approximation problem is: Given f continuous on X, find A^* minimizing $N(f - F(A, \cdot))$ over the set

$$P(X) = \{A : Q(A, x) \geqslant 0 \text{ for } x \in X, Q(A, \cdot) \not\equiv 0\}.$$

Such a parameter-value A^* is called best and $F(A^*, \cdot)$ is called a best approximation with respect to N.

The problem of the existence of best approximations is covered in [1].

DEFINITION. Q has the *zero-measure property* if $Q(A, \cdot) \not\equiv 0$ implies that the set of zeros of $Q(A, \cdot)$ is of measure zero.

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Since $R(\alpha A, x) = R(A, x)$ for all $\alpha > 0$, any rational which does not have the denominator vanishing identically can be normalized so that

$$\sum_{k=1}^{m} | a_{n+k} | = 1. \tag{0}$$

Compatible Norms

Definition. Let s be a subscript. We say that N_s is *compatible* with N if:

((i) There exists a finite set $\{M_1^s,..., M_p^s\}$ of measurable sets such that $M_i^s \cap M_j^s$ is empty for $i \neq j$ and $M_1^s \cup \cdots \cup M_p^s = X$.

(ii) There exists a corresponding set $X_s = \{x_1^s,..., x_p^s\}$ of points such that $x_i^s \in M_i^s$, $i = 1,..., p$.

(iii) For any function g on X, $N_s(g) = N(g_s)$, where we define

$$g_s(x) = g(x_i), \qquad x \in M_i^s, \qquad i = 1,..., p.$$

It is not difficult to see that any "τ-norm" on a finite subset of X has an equivalent compatible "norm." N_s could also come from a quadrature formula.

Definition. We say $\{N_k\} \to N$ if N_k is compatible with N, $k = 1,...,$ and

(iv) For any point x and neighborhood H of x, there is K such that for any $k > K$, there is a $y \in H$ with $g_k(x) = g(y)$.

Define $\rho_k(f) = \inf \{N_k(f - F(A, \cdot)) : A \in P(X_k)\}$.

Definition. *A is ϵ nearly best* with respect to N_k if $N_k(f - F(A, \cdot)) < \rho_k(f) + \epsilon$ and $A \in P(X_k)$.

Theorem. *Let Q have the zero-measure property and let bounded $F(B, \cdot)$ exist. Let $\tau(t) \to \infty$ as $| t | \to \infty$ and $| \sigma(t)| \to \infty$ as $t \to \infty$. Let neighborhoods be of positive measure. Let $N(f - F(B, \cdot)) < \infty$ imply that $f - F(B, \cdot)$ is Riemann integrable. Let $\{N_k\} \to N$, let A^k be ϵ_k nearly best with respect to N_k, and let $\epsilon_k \to 0$. Then $\{A^k\}$ has an accumulation point and any accumulation point is best with respect to N.*

Proof. Define $\| A \| = \max \{| a_j | : j = 1,..., n\}$. Suppose that $\{\| A^k \|\}$ is unbounded, then by taking a subsequence if necessary, we can assume that $\| A^k \| \to \infty$. By Lemma 2 of [1] there is a closed neighborhood G such that

$$\mu_k = \inf \{| f(x) - F(A^k, x)| : x \in G\} \to \infty.$$

There is a closed neighborhood H in G such that G is a neighborhood for each part of H. We have

$$N_k(f - F(A^k, \cdot)) \geqslant \int_H \tau(f - F(A^k, \cdot))]_k \geqslant \mu(H) \min \{\tau(y) : |y| \geqslant \mu_k\},$$

and the right-hand side tends to infinity. But

$$N_k(f - F(B, \cdot)) \leqslant \mu(X) \max \{\tau(u) : -\eta \leqslant u \leqslant \eta\},$$

where $\eta = \|f - F(B, \cdot)\|_\infty$. Hence near optimality of A^k is contradicted. We can, therefore, assume that $\{A^k\}$ is bounded and has an accumulation point A. Assume without loss of generality that $\{A^k\} \to A$. We claim that $Q(A, \cdot) \geqslant 0$. Suppose not, then there is $\epsilon > 0$ and $x \in X$ with $Q(A, x) < -\epsilon$. There is a closed neighborhood J of x such that $Q(A, y) < -\epsilon$ for $y \in J$. For all k sufficiently large, $Q(A^k, y) < -\epsilon/2$ for $y \in J$. Applying (iv), we see that $A^k \notin P(X_k)$ for all k sufficiently large and we have a contradiction.

We now prove that

$$N(f - F(A, \cdot)) \leqslant \limsup_{k \to \infty} N_k(f - F(A^k, \cdot)). \tag{1}$$

Let x not be a zero of $Q(A, \cdot)$ and $\epsilon > 0$ be given. We wish to prove that

$$|\tau(f(x) - F(A^k, x))]_k - \tau(f(x) - F(A, x))| < \epsilon \tag{2}$$

for all k sufficiently large. By continuity of τ there is $\nu > 0$ with

$$|\tau(w) - \tau(f(x) - F(A, x))| < \epsilon, \qquad |w - (f(x) - F(A, x))| < \nu. \tag{3}$$

There exists a neighborhood G of x such that $Q(A, y) > 0$ for $y \in G$, hence $R(A, \cdot)$ is continuous on G and $f - F(A, \cdot)$ is continuous into the extended real line on G. By arguments similar to the previous we can show that if $f - F(A, \cdot)$ attains an infinite value on G, then $N_k(f - F(A^k, \cdot)) \to \infty$, giving a contradiction. Hence $f - F(A, \cdot)$ is continuous on G. There is a closed neighborhood H of x contained in G such that

$$|(f(y) - F(A, y)) - (f(x) - F(A, x))| < \nu/2, \qquad y \in H. \tag{4}$$

Now $f - F(A^k, \cdot)$ converges uniformly to $f - F(A, \cdot)$ on H, so for all k sufficiently large

$$|(f(y) - F(A^k, y)) - (f(y) - F(A, y))| < \nu/2, \qquad y \in H.$$

By this and (4) we have

$$|(f(y) - F(A^k, y)) - (f(x) - F(A, x))| < \nu, \qquad y \in H.$$

By hypothesis (iv)

$$f(x) - F(A^k, x)]_k = f(y) - F(A^k, y), \qquad y \in H$$

and by (3), (2) is satisfied. It follows that $\tau(f - F(A^k, \cdot))]_k \to \tau(f - F(A, \cdot))$ on all points at which $Q(A, \cdot)$ does not vanish, so we have pointwise convergence almost everywhere. Further $N_k(f - F(A^k, \cdot))$ is uniformly bounded, so by Fatou's theorem, (1) holds.

Now suppose A is not best with respect to N. Then there is $B \in P(X)$ and $\epsilon > 0$ with

$$N(f - F(B, \cdot)) < N(f - F(A, \cdot)) - \epsilon.$$

We have

$$N_k(f - F(B, \cdot)) \to N(f - F(B, \cdot))$$

since $f - F(B, \cdot)$ is Riemann integrable.

Let $N_{k(j)}(f - F(A^{k(j)}, \cdot)) \to \limsup_{k\to\infty} N_k(f - F(A^k, \cdot))$; then for all j sufficiently large

$$N_{k(j)}(f - F(B, \cdot)) < N_{k(j)}(f - F(A^{k(j)}, \cdot)) - \epsilon/2,$$

contradicting $A^{k(j)}$ being $\epsilon_{k(j)}$ nearly best with respect to $N_{k(j)}$.

A parameter A is called *admissible* on X if $Q(A, x) > 0$ for $x \in X$.

Remark. Let a best parameter to f on X be admissible, then the theorem remains true if we approximate with respect to N_k with parameter set

$$\hat{P}(X_k) = \{A : Q(A, x) > 0, x \in X_k\}.$$

To establish the remark, we let B at the end of the proof of the previous theorem be admissible on X.

The remark does not imply that an accumulation point A need be admissible on X (see the example at the end of the paper).

COROLLARY 1. *Let the hypotheses of Theorem* 1 *hold. Let there exist a unique parameter A of best approximation to f with respect to N under the normalization* (0) *and $Q(A, \cdot) > 0$. Then $\{A^k\} \to A$ and $Q(A^k, \cdot) > 0$ for all k sufficiently large.*

If the hypotheses of Corollary 1 holds, there exists a best admissible approximation with respect to N_k for all k sufficiently large.

COROLLARY 2. *Let the hypotheses of Corollary* 1 *hold and σ be continuous on an open set containing the range of $R(A, \cdot)$. Then $\{F(A^k, \cdot)\}$ converges uniformly to $F(A, \cdot)$ and $N(f - F(A^k, \cdot)) \to N(f - F(A, \cdot))$.*

Without the uniqueness condition of Corollary 1, the conclusions of the above corollaries may not hold.

EXAMPLE. Let $X = [0, 1]$ and N be the L_p norm on $[0, 1]$, $p \geqslant 1$. Let N_k be based on evaluation at the points $\{1/k, 2/k, ..., (k-1)/k, 1\}$. Let $f = 0$. Let the approximations be a family of ordinary rational functions. There exist $\alpha_k > 0$ such that $N_k(-\alpha_k/x) < 1/k$, hence α_k/x is $1/k$ nearly best. However,

$$N(-\alpha_k/x) = \alpha_k \int_0^1 x^{-p}\, dx = \alpha_k \log(x)]_0^1 = \infty, \qquad p = 1$$
$$= \alpha_k x^{1-p}/(1-p)]_0^1 = \infty, \qquad p > 1.$$

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